Transient regime and superradiance in a short-pulse free-electron-laser oscillator

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We investigate the transient evolution of radiation in a low-gain free-electron-laser oscillator, driven by synchronized electron bunches much shorter than the slippage. We calculate analytically the radiation intensity and the gain in the linear regime. We show that, in the nonlinear regime and in the ideal case without losses, the radiation field is described, as in the short-pulse, high-gain amplifier, by the self-similar superradiant solution.

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Recently increasing interest has been given to free-electron-laser oscillators driven by electron bunches shorter than or equal to the slippage distance [1,2]. In particular, superradiant emission of short radiation pulses, whose peak intensity is proportional to the square of the electron beam density [3], has been observed numerically [4] for pulse length equal to the slippage distance. Superradiance has been supposed to play a role in several effects occurring in the short-pulse operation, for example, the observation of limit cycles in a desynchronized cavity, with a periodic generation of radiation pulses [5]. At present, however, superradiant emission in oscillators has not yet been demonstrated analytically. Moreover, theoretical works dealing with short-pulse propagation, apart from numerical simulations, have been focused only on the research of stationary solutions, called supermodes [6], and not on the transient regime.

In this paper, we calculate analytically the radiation field in the transient linear regime of an oscillator driven by ultrashort electron bunches, in a condition of perfect cavity synchronization. Although no stable solutions exist in this case, the small-signal gain per pass decreases as $n^{-2/3}$ as a function of the round-trip number n. Due to the slow decrease of the gain with n, if the losses are very low and the initial startup power sufficiently large, the radiation can reach the nonlinear regime for a sufficiently high value of power. We demonstrate that, in the case of negligible losses, the radiation intensity is superradiant and the field is described by the superradiant self-similar solution. A previous analysis [7] has shown that this solution describes the superradiant regime in a high-gain, short-pulse amplifier. The maximum of the superradiant pulse amplitude is proportional to n and its width decreases as $n^{-1/2}$. For large n, secondary peaks of lower intensity follow the principal peak.

We consider the usual model of equations in the dimensionless form and in the Compton approximation, for the complex field amplitude A and the electron phase θ [3]:

$$\frac{\partial A(z_1,z_2)}{\partial z_1} = f(z_1)\left\langle e^{-i\theta(z_1,z_2)}\right\rangle, \tag{1}$$

$$\frac{\partial^2\theta(z_1,z_2)}{\partial z_2^2} = -\left[A(z_1,z_2)e^{i\theta(z_1,z_2)} + \text{c.c.}\right]. \tag{2}$$

In these equations, $z_1 = (z - v_{\parallel}t)/L_c$, $z_2 = (ct - z)/L_c$, and $z_1 + z_2 = z/L_g$, where z is the distance along the axis of an undulator with period $\lambda_w = 2\pi/k_w$ and rms parameter a_w, $L_c = \lambda/4\pi\rho$ and $L_g = \lambda_w/4\pi\rho$ are the cooperation and the gain lengths, $\lambda = \lambda_w(1 + a_w^2)/2\gamma_0^2$ is the resonant wavelength, and $v_{\parallel}$ is the average longitudinal beam velocity; $\rho = (1/\gamma_0)(a_w\lambda_w F/4\pi r_b)^{2/3}(I/I_0)^{1/3}$ is the fundamental free-electron-laser parameter, F is 1 for a helical undulator and the well-known difference of Bessel functions for a linear undulator, $mc^2\gamma_0$ is the initial beam energy, r_b is the beam radius, I is the peak current, and $I_0 = 4\pi\epsilon_0 mc^3/e \sim 17\,000$ A is the Alfvén limit current (perfect transverse overlapping between the electron and radiation pulses has been assumed). In Eq. (1) the angular brackets indicate an average over the particles and $f(z_1)$ is the longitudinal electron profile normalized to 1 at the peak.

In an oscillator, the radiation is reflected backwards and then forwards for the next pass through the undulator, so that the input field for the $(n+1)$th pass is

$$A_0^{(n+1)}(z_2 - \delta) = rA_f^{(n)}(z_2), \tag{3}$$

where A_f and A_0 are the fields at the end and at the entrance of the undulator, r accounts for the reduction in amplitude due to energy losses at the mirrors, and $\delta = 2\Delta\mathcal{L}/L_c$ is the cavity detuning, where $\Delta\mathcal{L}$ is the cavity shortening relative to a perfect synchronism between the cavity round-trip time at the vacuum speed of light and the injection period of the electron micropulse. By shortening the cavity by $\Delta\mathcal{L}$, the optical pulse is pushed forward each pass by $\Delta z_2 = \delta$.

We limit the analysis to ultrashort electron micropulses, with $f(z_1) = \sigma\delta(z_1)$, where $\delta(z_1)$ is Dirac's delta function and σ is the beam length in units of L_c for a beam much shorter than the slippage, $\sigma \ll z/L_g$.

In order to obtain the solution in the linear regime, we have integrated Eq. (2) from $z = 0$ and we have substituted the solution in Eq. (1); then, we have expanded the right-hand side term of Eq. (1) in powers of the field amplitude A keeping into account only the linear terms. Assuming the initial phases θ_0 uniformly distributed over 2π, with $\langle\exp(im\theta_0)\rangle = 0$ for $m = 1, 2$, the solution can be obtained using the Laplace transform technique, as done, for example, in Ref. [7]. Using Eq. (3), the following result is obtained:

$$A_0^{(n+1)}(z_2-\delta) = rA_0^{(n)}(z_2) + \eta r \sum_{k=0}^{\infty} \frac{(i\sigma)^{k+1}}{(k+1)!(2k+1)!} \times \int_0^{z_2} d\xi (z_2-\xi)^{2k+1} A_0^{(n)}(\xi), \quad (4)$$

where $\eta = 1$ for $0 < z_2 < \bar{z} = L/L_g = 4\pi\rho\ N_w$ where $L = \lambda_w N_w$ is the undulator length, and $\eta = 0$ elsewhere. This equation gives the linear solution for an arbitrary value of gain per pass. We restrict our analysis to the low-gain regime, for $\sigma\bar{z}^2 \ll 1$. In this limit, only the first term $k = 0$ in the sum of Eq. (4) can be retained. Moreover, we assume small losses, $r = 1 - \alpha/2$, with $\alpha \ll 1$, and small cavity detuning, $\delta \ll \bar{z}$. Under these assumptions, we expand $A(z_2 - \delta)$ in a Taylor series retaining only the first linear term proportional to the first derivative with respect to z_2 and we treat the pass number n as a continuous variable, $\tau = n$. Then, Eq. (4) can be approximated by the following differential equation for $A(z_2,\tau) = A_0^{(n)}(z_2)$:

$$\frac{\partial A(z_2,\tau)}{\partial \tau} - \delta\, \frac{\partial A(z_2,\tau)}{\partial z_2} + \frac{\alpha}{2} A(z_2,\tau) = i\sigma\eta \int_0^{z_2} d\xi (z_2-\xi) A(\xi,\tau). \quad (5)$$

This equation describes the linear evolution of the radiation in the cavity in the low-gain, short-pulse regime.

With the same low-gain approximations used to derive Eq. (5), we can obtain the equations describing the nonlinear evolution in the low-gain regime substituting $A(z_1, z_2)$ for the input field $A_0(z_2)$ in the equation for the electron phases (2). Equations (1)–(3) are then approximated by the following equations:

$$\frac{\partial A(z_2,\tau)}{\partial \tau} - \delta\, \frac{\partial A(z_2,\tau)}{\partial z_2} + \frac{\alpha}{2} A(z_2,\tau) = \sigma\eta \left\langle \exp[-i\tilde{\theta}(z_2,\tau)] \right\rangle, \quad (6)$$

$$\frac{\partial^2 \tilde{\theta}(z_2,\tau)}{\partial z_2^2} = -\left\{ A(z_2,\tau) \exp[i\tilde{\theta}(z_2,\tau)] + \text{c.c.} \right\}, \quad (7)$$

where $\tilde{\theta}(z_2,\tau) = \theta(z_1 = 0, z_2)$. It is easy to verify that Eq. (5) can be obtained directly by linearizing Eqs. (6) and (7).

We observe that eqs. (6) and (7) have the same structure as the single-pass equations (1) and (2), with two important differences: (a) the radiation interacts with the electrons over the slippage length, $0 < z_2 < \bar{z}$, whereas in the single-pass model, the radiation interacts with the electrons over the electron beam length $(0 < z_1 < L_b/L_c)$; (b) due to the cavity shortening δ, the radiation moves forward or backward in z_2 depending on whether δ is negative or positive, respectively. In the first case, a boundary condition for A must be assigned to the leading edge $z_2 = 0$, and the radiation is shifted by $|\delta|\tau$ along the positive direction of z_2, leaving the interaction region from the trailing edge $z_2 = \bar{z}$; in this case the radiation propagates in the same direction as the electrons. In the second case, when δ is positive (cavity shorter than perfect synchronism), a boundary condition for A must be assigned to the trailing edge $z_2 = \bar{z}$, and the radiation propagates backward, leaving the interaction region from the leading edge $z_2 = 0$ and moving towards the electrons. This negative slippage provides a positive feedback with bunching and power carried by the electrons and radiation in counterpropagating directions, leading to self-modulation of the emitted intensity [8]. A positive detuning δ allows for the existence in the linear regime of stationary solutions of the form $A(z_2,\tau) = \exp(\mu\tau)A_s(z_2)$, where μ is the complex eigenvalue with positive real part [6]. These solutions represent stable pulse configurations whose intensity is multiplied each pass by a constant gain factor.

Although no stable configurations exist in the linear regime for exact synchronism [6], it is important to investigate the transient regime in this case. We assume in the rest of the analysis $\delta = 0$, reserving the analysis of the more complex case with positive cavity detuning δ to a further publication. We demonstrate in the following that for $\delta = 0$ and negligible cavity losses the radiation emitted is superradiant.

We start obtaining the linear solution for $\delta = 0$. Assuming a uniform initial excitation, $A(z_2, \tau = 0) = A_0$, Eq. (5) has the following solution:

$$A(z_2,\tau) = A_0 e^{-\alpha\tau/2} \sum_{m=0}^{\infty} \frac{i^m y^{2m}}{m!(2m)!} \simeq (A_0/2\sqrt{3\pi})(2/y)^{1/3} \times e^{(3/2)(\sqrt{3}+i)(y/2)^{2/3} - i\pi/12 - \alpha\tau/2}, \quad (8)$$

where $y = \sqrt{\sigma\tau}\, z_2$, and the last approximated expression in Eq. (8) is valid for large values of y. We observe that, apart from the factor $\exp(-\alpha\tau/2)$, the field depends only on the self-similar variable y and has the same form as the linear solution for a high-gain amplifier starting from noise and in the limit of large slippage [7], where in that case $y = \sqrt{z_1}\, z_2$. In Fig. 1 we plot the radiation profile vs z_2, for $\tau = 300$, $\bar{z} = 2$, $\sigma = 0.1$, $\alpha = 0$, and $A_0 = 0.01$, as given by Eq. (8). We observe that the field is an increasing function of z_2, with a maximum in the trailing edge $z_2 = \bar{z}$, showing the well-known lethargy effect for which the effective radiation group velocity appears to be smaller than the vacuum speed of light. In the present case, with ultrashort micropulses, the radiation intensity peak moves at the electron velocity.

By integrating the intensity $|A|^2$ over z_2, we calculate the radiation energy:

$$\mathcal{E}(\tau) = \int_0^{\bar{z}} d\xi |A(\xi,\tau)|^2 \simeq \frac{A_0^2}{12\pi\sqrt{\sigma\tau}} e^{-\alpha\tau} \int_0^{\sqrt{\sigma\tau}\ \bar{z}} dy (2/y)^{2/3} e^{3\sqrt{3}(y/2)^{2/3}}. \quad (9)$$

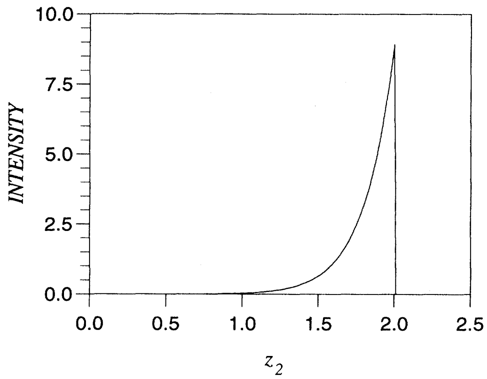

FIG. 1. Linear regime: intensity $|A|^2$ vs z_2 for $\tau = 300$, $\bar{z} = 2$, $\sigma = 0.1$, $\alpha = 0$, and $A_0 = 0.01$, as given by Eq. (8).

We assumed that $\mathcal{E}$ receives its main contribution from the region in which the asymptotic form of A is valid. We next crudely evaluate Eq. (9) for the case in which the exponent takes on its maximum value at the upper limit, expanding the exponent about its upper limit. This approximation yields

$$\mathcal{E}(\tau) = (A_0^2/12\pi\sqrt{3})(2/\sigma^2\tau^2\,\bar{z})^{1/3} \times \exp[3\sqrt{3}(\sqrt{\sigma\tau}\,\bar{z}/2)^{2/3} - \alpha\tau]. \tag{10}$$

Equation (10) requires for its validity $(\sqrt{\sigma\tau}\,\bar{z})^{2/3} \gg 1$. From Eq. (10), we calculate the gain per pass:

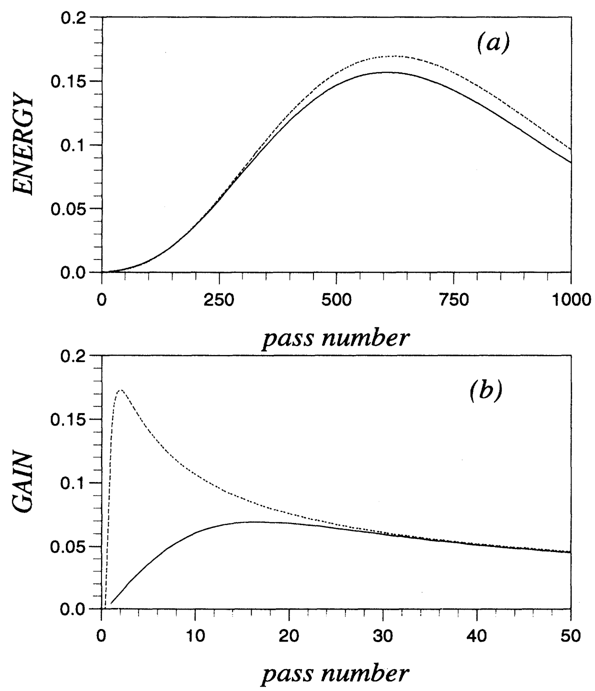

FIG. 2. Linear regime: Energy $\mathcal{E}$ (a) and gain $\mathcal{G}$ (b) vs pass number τ, for $\alpha = 0.01$, from a multipass numerical simulation (continuous line) and from Eqs. (10) and (11) (dashed line); the other parameters are the same as in Fig. 1.

$$\mathcal{G}(\tau) = \frac{d}{d\tau}[\ln\mathcal{E}(\tau)] = \sqrt{3}\left[\frac{\sqrt{\sigma}z}{2\tau}\right]^{2/3} - \frac{2}{3\tau} - \alpha. \tag{11}$$

We observe that $\mathcal{G}$ decreases as $\tau^{-2/3}$ for τ large; hence there is no stable gain for exact cavity synchronism, although the gain decreases rather slowly. In Figs. 2(a) and 2(b) we compare the energy and the gain as calculated from Eqs. (10) and (11) (dashed lines) with the result of a multipass simulation, integrating Eqs. (1)–(3), for a rectangular electron beam profile of length $\sigma = 0.1$ and for $\bar{z} = 2$, $\delta = 0$, $A_0 = 0.01$, and $\alpha = 0.01$ (continuous line). We observe that although the asymptotic expression (11) gives a higher maximum than the exact result of the simulation, it describes very accurately the gain for $\tau > 30$. By making comparisions between the exact solution for the gain, obtained by numerical multipass simulations, with the approximated expression, it is found that the estimate of $\mathcal{G}$ obtained from Eq. (11) is accurate to within 3% for values of the ratio between the slippage and the beam length $\bar{z}/\sigma = \lambda N_w/L_b \geq 4$.

From the results of the linear analysis we have seen that, for $\alpha = 0$, the solution $A(z_2,\tau)$ is a function of the self-similar variable $y = \sqrt{\sigma\tau}\,z_2$. On the basis of the results obtained for the nonlinear regime of superradiance in high-gain, short-pulse amplifiers [7], one can expect that an analogous superradiant emission should take place in the low-gain, short-pulse oscillator. In fact, Eqs. (6) and (7) admit, for negligible cavity losses $\alpha = 0$, the following self-similar solution in the general case of arbitrary cavity detuning:

$$A(z_2,\tau) = \sigma\,\tau\,A_1(y), \tag{12}$$

$$\tilde{\theta}(z_2,\tau) = \theta_1(y), \tag{13}$$

where $y = \sqrt{\sigma\tau}(z_2 + \delta\tau)$ and $A_1(y)$ and $\theta_1(y)$ satisfy the following ordinary differential equations:

$$\frac{y}{2}\frac{dA_1}{dy} + A_1 = \langle e^{-i\theta_1}\rangle \tag{14}$$

$$\frac{d^2\theta_1}{dy^2} = -\left[A_1 e^{i\theta_1} + \text{c.c.}\right]. \tag{15}$$

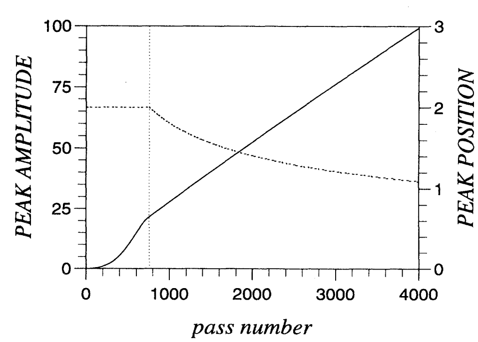

FIG. 3. Nonlinear regime: peak amplitude $|A|_{peak}$ (continuous line) and peak position z_{peak} (dashed line) vs pass number τ from the numerical multipass simulation, with the same parameters as in Fig. 1 and without losses, $r = 1$. For $\tau < 760$ (vertical line) the system is still in the linear regime.

In order to verify the existence of this asymptotic solution, we have integrated numerically Eqs. (1)–(3) for $\delta = 0$, $r = 1$, $\sigma = 0.1$, $\overline{z} = 2$, and $A_0 = 0.01$. In Fig. 3 we plot the peak amplitude $|A|_{peak}$ (continuous line) and the peak position z_{peak} (dashed line) as a function of the round-trip number τ. In the linear regime, the peak is at the trailing edge $z_2 = 2$. For $\tau > 760$ (vertical dotted line) the radiation leaves the linear regime and the peak moves towards the leading edge $z_2 = 0$; we observe that the amplitude becomes proportional to τ, in agreement with the self-similar solution (12). In Fig. 4 we make a comparision between the exact radiation profile at $\tau = 4000$ (continuous line), as it results from the numerical simulation for the same parameters as in Fig. 3, and the solution (12), where $A_1(y)$ has been obtained solving numerically Eqs. (14) and (15) for $A_1(0) = 5.75 \times 10^{-4}$ and $z_2 = y/\sqrt{\sigma\tau} + 0.2545$. The radiation profile follows very accurately the first peak of the superradiant solution (12) and somewhat less accurately the secondary peak, probably beacause of the finite length of the beam used in the numerical simulation.

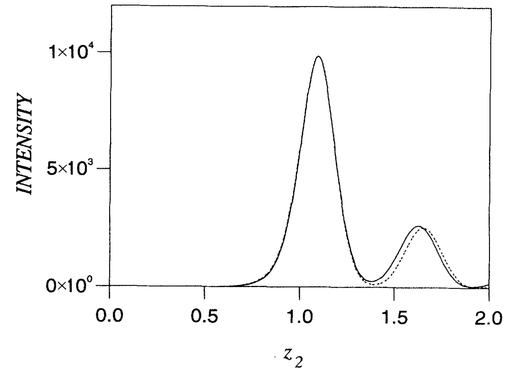

FIG. 4. Nonlinear regime: intensity $|A|^2$ vs z_2 for $r = 1$ and $\tau = 4000$ (continuous line), compared with the self-similar solution (12) (dashed line) with $A_1(0) = 5.75 \times 10^{-4}$; the other parameters are the same as in Fig. 1.

This result shows that for a perfectly synchronized cavity and in the limit of very short micropulses, the emission is superradiant. The superradiant scaling of the intensity follows from Eq. (12) and from the definition of A [3]: since $A \propto E_0/\sqrt{\rho I}$ and $\sigma \propto \rho L_b$, where L_b is the beam length, then $|E_0|^2 \propto \rho^3 I L_b^2 \propto I^2 L_b^2$ and the intensity is proportional to the square of the micropulse charge. The self-similar solution shows that the electron pulse generates a train of decreasing pulses with intensity growing as τ^2 and narrowing as $\tau^{-1/2}$, separated by a distance of about $10L_c/\sqrt{\sigma\tau}$ inside the slippage length λN_w. The pulse profile continues to narrow for an increasing number of passes through the cavity, so that no stable configuration can be reached. The continuous evolution of the radiation profile explains why, in general, when cavity losses are taken into account, the radiation energy decreases after the gain per pass has crossed the value imposed by the cavity losses.

In conclusion, we have demonstrated that in a synchronized oscillator driven by micropulses shorter than the slippage and in the ideal case without losses, the emission is superradiant and the radiation pulse has a self-similar profile, given by a train of peaks with decreasing intensity, whose distance is inversely proportional to the square root of the pass number. Further analytical work and numerical simulations are in progress to examine the relation between the superradiant solution and the limit-cycle behavior observed in desynchronized short-pulse oscillators [5].

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